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ABSTRACT

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A FACTOR ROTATION SCHEME GIVING ORTHOGONAL BOUNDS

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A Description of "Bounding Hyperplanes"

One primitive quality which is sought in factor rotation is that the planes which are defined by factor vectors envelop test vectors much as a multidimensional bunch of roses would be enveloped by florists' paper. L. L. Thurstone (1935) expressed this idea in two definitions.

> Definition: The orthogonal hyperplanes which bound a positive region in r dimensions will be called the orthogonal positive manifold.
>
> Definition: If the factorial matrix of the traits which are contained in a positive hyperplane is of rank (r-1), then the hyperplane is a bounding hyperplane or a positive coordinate hyperplane. (Thurstone, 1935)

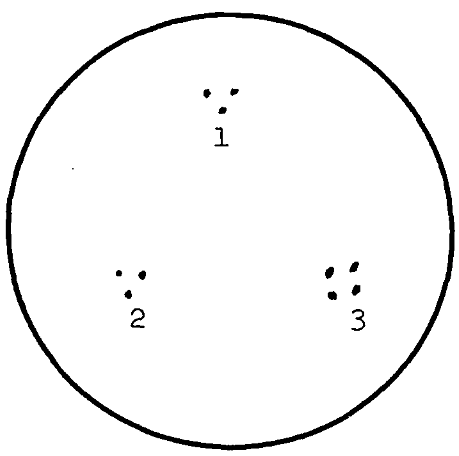

Figure 1

Graphically, if we had 10 tests which fell neatly into three clusters we could represent them on the surface of a sphere as in Figure 1.

The three clusters 1, 2, and 3 are neatly separated and hyperplane boundaries can be easily slipped around them to form a neat envelope, $\alpha\beta\gamma$.

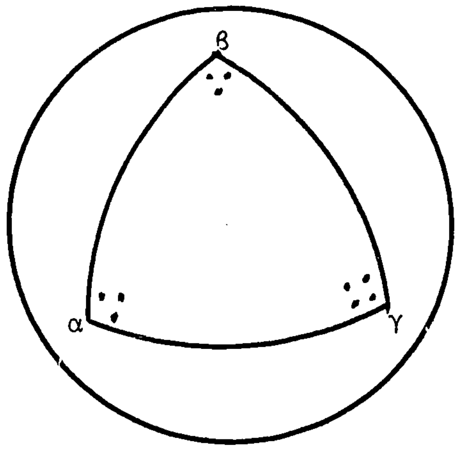

Figure 2

Unfortunately, real factor analytic studies are not neatly clustered and the mathematical techniques for locating the three vectors α, β, and γ are not so easy whether or not one requires orthogonality among the hyperplanes.

Relationship to Multiple Group Factor Analysis

The multiple group factor analysis technique (Harman, 1967, p. 241) is often used in simple factor analysis studies. With such data as proposed above, three factors a, b, and c could be isolated and graphed on the surface of a sphere as in Figure 3.

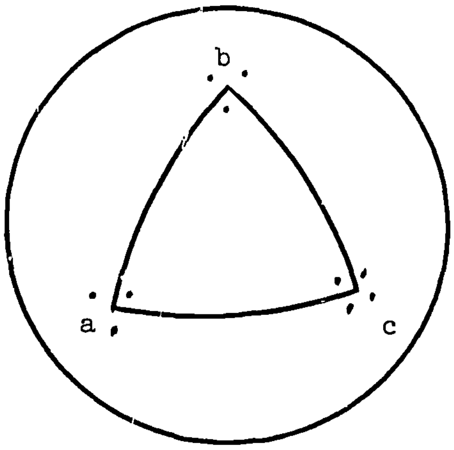

Figure 3

The factors are at the centroids of the three clusters. The hyperplane abc is not a bounding hyperplane and is not often orthogonal. But the method has the advantages of involving easy arithmetic and being a reasonable representation of reality.

Sometimes when there is a previous analysis of the data, the following calculation technique is used. Suppose $R=FF'$ is an orthogonal decomposition of a correlation matrix R (with perhaps a few small factors discarded). The matrix F has p rows for the variables and r columns for the factors. First one decides which variables are to form the first cluster, then one adds the row entries of these variables to form a new vector, say t_1, and reduces t_1 to unit length. Next, one picks the second cluster, adds the appropriate rows of F to form a vector t_2 and reduces t_2 to unit length, and so on for the r clusters.

The column vectors t_i are united to form a matrix T. The correlations among the rotated factors are calculated as

$$D^{-1}(T^{-1}T'^{-1})\ D^{-1} = \phi^{-1}$$

or

$$D^{-1}(T'T)^{-1}\ D^{-1} = \phi^{-1}$$

where D^{-1} is a diagonal matrix of the reciprocals of the square roots of the diagonals of $(T'T)^{-1}$. The proper rotation matrix for F is TD and the following equation holds:

$$R \approx FF' = [FTD]\ [D^{-1}(T'T)^{-1}D^{-1}]\ [DT'F']$$
$$= G\phi^{-1}G'\ ,\ \text{where } FTD = G\ .$$

The entries of G are still correlations between factors and variables. Occasionally when it appears that one variable in a cluster should carry more weight than others, that variable is added in two or three times to bring the centroid closer to it. In fact there is no reason why any arbitrary linear combination of rows of F cannot be used to obtain a rotation vector.

A Scheme of Kaiser's

Recently, Kaiser (1967) discussed the factor analytic relevance of a mathematical matrix decomposition called the "square root."

Here, if a correlation matrix R is decomposed into characteristic roots, Λ, and vectors, Q, so that $R = Q\Lambda Q'$ the square root of R is $A = Q\Lambda^{1/2}Q'$ and $R = AA$. Since A is symmetric, it is also true that $R = AA'$.

The factors, say S, which emerge from this decomposition are uncorrelated (orthogonal) and maximally correlated pairwise with the original variables. That is, they form an orthogonal bounding hyperplane.

The Method

This technique can be applied to ϕ^{-1}. Let H be factors derived from a multiple group technique, $HH' = \phi^{-1}$. Decompose ϕ^{-1} into AA where A is the square root of ϕ^{-1} with new factors S, where $SS' = I$.

$$HH' = \phi^{-1} = Q\Lambda^{1/2}Q'Q\,\Lambda^{1/2}Q' = ASS'A.$$

Or in terms of R

$$\begin{aligned} R \approx FF' &= [FTD]\,[D^{-1}(TT')^{-1}D^{-1}]\,[DT'F] \\ &= [FTD]\,\phi^{-1}[DT'F'] \\ &= [FTDA]\,SS'\,[ADT'F'] . \end{aligned}$$

The entries of FTDA are correlations between the p variables and the factors, but the factors determine the orthogonal bounding hyperplanes of best fit to predetermined centroids. The projection of test clusters and reference vectors α, β, and γ onto a sphere looks like Figure 2. When there is an occasional negative correlation between tests, it may be that a few test vectors lie outside the hyperplane $\alpha\beta\gamma$ and will have small negative correlations with some of the factors.

Regression weights for total variance solutions can be calculated in the usual manner when $R = XX'$ and $FTDA = G$, $X = GS$. The regression coefficients, β, are found to be $(G'G)^{-1}G'$.

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